Find the general solution of $x^2y'' + xy' + 4y = \csc(2\ln x)$. $\longrightarrow y'' + \frac{1}{x}y' + \frac{4}{x^2}y = \frac{\csc(2\ln x)}{x^2} = g(x)$ **SCORE: _____ / 10 PTS**

$r^2 + 4 = 0$

$r = \pm 2i$

2 $\underline{y_1 = \cos(2\ln x),\ y_2 = \sin(2\ln x)}$

$$W = \begin{vmatrix} \cos(2\ln x) & \sin(2\ln x) \\ -\frac{2}{x}\sin(2\ln x) & \frac{2}{x}\cos(2\ln x) \end{vmatrix} = \underline{\frac{2}{x}}\ 1$$

$$y_p = -\cos(2\ln x)\int \frac{\frac{\csc(2\ln x)}{x^2}\sin(2\ln x)}{\frac{2}{x}}dx + \sin(2\ln x)\int \frac{\frac{\csc(2\ln x)}{x^2}\cos(2\ln x)}{\frac{2}{x}}dx$$

$u = \sin(2\ln x)$

$du = \frac{2}{x}\cos(2\ln x)\,dx$

$$= \underline{-\cos(2\ln x)\int \frac{1}{2x}dx}\ 1 + \underline{\sin(2\ln x)\int \frac{\cos(2\ln x)}{2x\sin(2\ln x)}dx}\ 1\tfrac{1}{2}$$

$$= -\cos(2\ln x)\underline{\left(\tfrac{1}{2}\ln|x|\right)}\ \tfrac{1}{2} + \sin(2\ln x)\underline{\left(\tfrac{1}{4}\ln|\sin(2\ln x)|\right)}\ 1$$

$$y = 2\ \underline{-\tfrac{1}{2}\cos(2\ln x)\ln|x| + \tfrac{1}{4}\sin(2\ln x)\ln|\sin(2\ln x)|}$$

$$+\ \underline{c_1\cos(2\ln x) + c_2\sin(2\ln x)}\ 1$$

BONUS QUESTION: Find a second order linear differential equation whose general solution is **SCORE: _____ / 3 PTS**

$y = \underbrace{Ax^2 + Bx^2 \ln x}_{} + \underbrace{x^3}_{}$.

CAUCHY-EULER y_h $\quad$ y_p

$r = 2, 2$

$(r-2)^2 = 0$

$r^2 - 4r + 4 = 0$

1 $x^2 y'' - 3xy' + 4y = b(x)$

1 $x^2(6x) - 3x(3x^2) + 4(x^3) = x^3$

1 $x^2 y'' - 3xy' + 4y = x^3$

Find the general solution of $y'' - 5y' - 6y = xe^{-x}$. **SCORE: _____ / 14 PTS**

UNDETERMINED COEFFICIENTS SOLUTION

$$r^2 - 5r - 6 = 0$$

$$(r-6)(r+1) = 0$$

$$r = 6, -1$$

$$y_h = c_1 e^{6x} + c_2 e^{-x} \quad 2$$

$$y_p = x(Ax+B)e^{-x} = (Ax^2 + Bx)e^{-x} \quad 2$$

$$y_p' = (2Ax + B)e^{-x} + (-Ax^2 - Bx)e^{-x} = (-Ax^2 + (2A-B)x + B)e^{-x} \quad 2$$

$$y_p'' = (-2Ax + (2A-B))e^{-x} + (Ax^2 + (-2A+B)x - B))e^{-x}$$

$$y_p'' = (Ax^2 + (-4A+B)x + (2A-2B))e^{-x} \quad 2$$

$$-5y_p' \quad +(5A^2 + (-10A+5B)x \quad -5B)e^{-x}$$

$$-6y_p \quad +(-6Ax^2 \quad -6Bx \quad)e^{-x}$$

$$= \quad 2 \quad (-14Ax + (2A-7B))e^{-x} = xe^{-x}$$

$$-14A = 1 \rightarrow A = -\frac{1}{14} \quad 1$$

$$2A - 7B = 0 \rightarrow B = \frac{2}{7}A = -\frac{1}{49} \quad 1$$

$$y = 1 \quad \left(-\frac{1}{14}x^2 - \frac{1}{49}x\right)e^{-x} + c_1 e^{6x} + c_2 e^{-x} \quad 1$$

OR

VARIATION OF PARAMETERS SOLUTION

$$\underbrace{y_1 = e^{-x} \qquad y_2 = e^{6x}}_{} \quad 2$$

$$W = \begin{vmatrix} e^{-x} & e^{6x} \\ -e^{-x} & 6e^{6x} \end{vmatrix} = \underbrace{7e^{5x}}_{} \quad 2$$

$$y_p = -e^{-x}\int \frac{xe^{-x}\cdot e^{6x}}{7e^{5x}}\,dx + e^{6x}\int \frac{xe^{-x}\cdot e^{-x}}{7e^{5x}}\,dx$$

$$= \underbrace{-\frac{1}{7}e^{-x}\int x\,dx}_{2} + \underbrace{\frac{1}{7}e^{6x}\int xe^{-7x}\,dx}_{2}$$

$$= -\frac{1}{7}e^{-x}\underbrace{\left(\frac{1}{2}x^2\right)}_{1} + \frac{1}{7}e^{6x}\underbrace{\left(-\frac{1}{7}xe^{-7x} - \frac{1}{49}e^{-7x}\right)}_{2}$$

u		dv
x		e^{-7x}
1	$+$	$-\frac{1}{7}e^{-7x}$
0	$-$	$\frac{1}{49}e^{-7x}$

$$= -\frac{1}{14}x^2e^{-x} - \frac{1}{49}xe^{-x} + Ce^{-x} \quad 2$$

$$y = \underbrace{-\frac{1}{14}x^2e^{-x} - \frac{1}{49}xe^{-x}}_{2} + \underbrace{c_1e^{6x} + c_2e^{-x}}_{1}$$

Use the annihilator method to find the <u>form of a particular solution</u> of $y'' + 4y' + 5y = 6x^2 - 3e^{-2x}\sin x$. **SCORE: _____ / 6 PTS**

<u>You MUST show the usage of the annihilator. Do NOT solve for the coefficients in the particular solution.</u>

D^3 ANNIHILATES $6x^2$

$(D+2)^2+1$ ANNIHILATES $-3e^{-2x}\sin x$

$r^2+4r+5=0$

$r=-2\pm i$

1 $y_h = C_1e^{-2x}\cos x + C_2e^{-2x}\sin x$

1 1

$D^3\left((D+2)^2+1\right)(D^2+4D+5)[y] = D^3\left((D+2)^2+1\right)[6x^2-3e^{-2x}\sin x]$

$D^3\left((D+2)^2+1\right)^2[y]=0 \rightarrow r=0,0,0,-2\pm i,-2\pm i$ $\frac{1}{2}$

$y = A+Bx+Cx^2 + \underbrace{De^{-2x}\cos x + Ee^{-2x}\sin x}_{y_h} + Fxe^{-2x}\cos x + Gxe^{-2x}\sin x$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$y_p = A+Bx+Cx^2 + Fxe^{-2x}\cos x + Gxe^{-2x}\sin x$ 1